

# Exact Computation of Minimum Sample Size for Estimating Proportion of Finite Population \*

Xinjia Chen

July 2007

## Abstract

In this paper, we develop an exact method for the determination of the minimum sample size for estimating the proportion of a finite population with prescribed margin of error and confidence level. By characterizing the behavior of the coverage probability with respect to the proportion, we show that the computational complexity can be significantly reduced and bounded regardless of population size.

## 1 Introduction

The estimation of the proportion of a finite population is a basic and very important problem in probability and statistics [3, 4]. The problem is formulated as follows.

Consider a finite population of  $N$  units, among which there are  $M$  units having a certain attribute. It is a frequent problem to estimate the proportion  $\frac{M}{N}$  by sampling without replacement. Let  $n$  be the sample size and  $\mathbf{k}$  be the number of units that found to carry the attribute. The estimate of the proportion is taken as  $\frac{\mathbf{k}}{n}$ . A crucial question in the estimation is as follows:

*Given the knowledge that  $M$  belongs to interval  $[L, U]$ , what is the minimum sample size  $n$  that guarantees the difference between  $\frac{\mathbf{k}}{n}$  and  $\frac{M}{N}$  be bounded within some prescribed margin of error with a confidence level higher than a prescribed value?*

Conventionally, the exact method requires evaluation of the coverage probability for all values of  $M$  in  $[L, U]$  for sample sizes incrementing from 2 to a number large enough. Since the range of interval  $[L, U]$  can be as wide as  $[0, N]$ , the number of evaluations of coverage probability can be very large if the population size  $N$  is large. The main contribution of this paper is to provide exact method for the computation of minimum sample size such that the total number of evaluations of

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\*The author had been previously working with Louisiana State University at Baton Rouge, LA 70803, USA, and is now with Department of Electrical Engineering, Southern University and A&M College, Baton Rouge, LA 70813, USA; Email: chenxinjia@gmail.com

coverage probability can be significantly reduced and bounded regardless of the population size  $N$ . Specially, we demonstrate that a small subset of the integers in interval  $[L, U]$  needs to be evaluated.

The paper is organized as follows. In Section 2, the techniques for computing the minimum sample size is developed with the margin of error taken as a bound of absolute error. In Section 3, we derive corresponding sample size method by using relative error bound as the margin of error. In Section 4, we develop techniques for computing minimum sample size with a mixed error criterion. Section 5 is the conclusion. The proofs are given in Appendices.

Throughout this paper, we shall use the following notations. The set of integers is denoted by  $\mathbb{Z}$ . The ceiling function and floor function are denoted respectively by  $\lceil \cdot \rceil$  and  $\lfloor \cdot \rfloor$  (i.e.,  $\lceil x \rceil$  represents the smallest integer no less than  $x$ ;  $\lfloor x \rfloor$  represents the largest integer no greater than  $x$ ). For non-negative integer  $m$ , the combinatoric function  $\binom{m}{z}$  with respect to integer  $z$  means

$$\binom{m}{z} = \begin{cases} \frac{m!}{z!(m-z)!} & \text{for } 0 \leq z \leq m, \\ 0 & \text{for } z < 0 \text{ or } z > m. \end{cases}$$

We denote

$$S(n, k, l, M, N) = \sum_{i=k}^l \frac{\binom{M}{i} \binom{N-M}{n-i}}{\binom{N}{n}}.$$

The notation “ $\Longleftrightarrow$ ” means “if and only if”. The other notations will be made clear as we proceed.

## 2 Control of Absolute Error

For  $\varepsilon, \delta \in (0, 1)$ , it is desirable in many situations to find the minimum sample size  $n$  such that

$$\Pr \left\{ \left| \frac{\mathbf{k}}{n} - \frac{M}{N} \right| < \varepsilon \right\} > 1 - \delta$$

for any  $M$  in interval  $[L, U]$ . Here the interval  $[L, U]$  is introduced to take into account the knowledge of  $M$ . If no information about  $M$  is available,  $[L, U]$  is taken as  $[0, N]$ . The quantity  $\Pr \left\{ \left| \frac{\mathbf{k}}{n} - \frac{M}{N} \right| < \varepsilon \right\}$  is referred to as the coverage probability. An essential step to find the minimum sample size is to determine whether a fixed sample size  $n$  is large enough to ensure that the coverage probability is above  $1 - \delta$  for any  $M$  in  $[L, U]$ . By the conventional method, for a fixed sample size  $n$ , the total number of evaluations of the coverage probability is  $U - L + 1$ . The computation can be high for large population size  $N$ . Interestingly, we discovered that the number of evaluations of the coverage probability can be significantly reduced by taking advantage of the coverage property as characterized by Theorem 1 at below.

**Theorem 1** *Let  $N, n$  and  $\varepsilon$  be fixed. Let  $L$  and  $U$  be two integers such that  $0 \leq L \leq U \leq N$ . Suppose that  $L \leq M \leq U$ . Then, the minimum of  $\Pr \left\{ \left| \frac{\mathbf{k}}{n} - \frac{M}{N} \right| < \varepsilon \right\}$  with respect to  $M$  is attained at  $\{L, U\} \cup \left\{ \left\lfloor N \left( \frac{k}{n} - \varepsilon \right) \right\rfloor \in (L, U) : k \in \mathbb{Z} \right\} \cup \left\{ \left\lceil N \left( \frac{k}{n} + \varepsilon \right) \right\rceil \in (L, U) : k \in \mathbb{Z} \right\}$ , which has less than  $2n \left( \frac{U-L-1}{N} \right) + 4$  elements.*

See Appendix A for a proof.

By the fact of symmetry that  $\Pr\left\{\left|\frac{\mathbf{k}}{n} - \frac{M}{N}\right|\right\} = \Pr\left\{\left|\frac{n-\mathbf{k}}{n} - \frac{N-M}{N}\right|\right\}$ , we can restrict  $M$  to be no larger than  $\lceil \frac{N}{2} \rceil$ . Hence, without loss of generality, we can assume that  $0 \leq L < U \leq \lceil \frac{N}{2} \rceil$ . Specially, for  $L = 0$ ,  $U = \lceil \frac{N}{2} \rceil$ , the total number of evaluations of coverage probability is less than

$$2n \left( \frac{\lceil \frac{N}{2} \rceil - 1}{N} \right) + 3 < 2n \left( \frac{\frac{N}{2}}{N} \right) + 3 = n + 3$$

since the coverage probability for  $L = 0$  is 1. This means that, in the situation that no information about  $M$  is available, the total number of evaluations of coverage probability is at most  $\boxed{n + 2}$ , which is independent of the population size  $N$ .

### 3 Control of Relative Error

For  $\varepsilon, \delta \in (0, 1)$ , it is interesting to find the minimum sample size such that

$$\Pr \left\{ \left| \frac{\mathbf{k}}{n} - \frac{M}{N} \right| < \varepsilon \frac{M}{N} \right\} > 1 - \delta$$

for any  $M$  in the interval  $[L, U]$ . For the purpose of reducing the number of evaluations of the coverage probability, we have

**Theorem 2** *Let  $N, n$  and  $\varepsilon$  be fixed. Let  $L$  and  $U$  be two integers such that  $0 \leq L \leq U \leq N$ . Suppose that  $L \leq M \leq U$ . Then, the minimum of  $\Pr \left\{ \left| \frac{\mathbf{k}}{n} - \frac{M}{N} \right| < \varepsilon \frac{M}{N} \right\}$  with respect to  $M$  is attained at*

$$\{L, U\} \cup \left\{ \left\lfloor \frac{Nk}{(1+\varepsilon)n} \right\rfloor \in (L, U) : k \in \mathbb{Z} \right\} \cup \left\{ \left\lceil \frac{Nk}{(1-\varepsilon)n} \right\rceil \in (L, U) : k \in \mathbb{Z} \right\},$$

which has less than  $2n \left( \frac{U-L-1}{N} \right) + 4$  elements.

We omit the proof of Theorem 2 because of its similarity to that of Theorem 1.

### 4 Control of Absolute Error or Relative Error

Let  $\varepsilon_a \in (0, 1)$  and  $\varepsilon_r \in (0, 1)$  be respectively the margins of absolute and relative error. Let  $\delta \in (0, 1)$ , it is frequently useful to find the minimum sample size such that

$$\Pr \left\{ \left| \frac{\mathbf{k}}{n} - \frac{M}{N} \right| < \varepsilon_a \text{ or } \left| \frac{\mathbf{k}}{n} - \frac{M}{N} \right| < \varepsilon_r \frac{M}{N} \right\} > 1 - \delta$$

for any  $M$  in the interval  $[L, U]$ . To reduce the computational complexity, we have

**Theorem 3** Let  $N$ ,  $n$ ,  $\varepsilon_a$  and  $\varepsilon_r$  be fixed. Let  $L$  and  $U$  be two integers such that  $0 \leq L < \frac{N\varepsilon_a}{\varepsilon_r} < U \leq N$ . Suppose that  $L \leq M \leq U$ . Then, the minimum of  $\Pr \left\{ \left| \frac{\mathbf{k}}{n} - \frac{M}{N} \right| < \varepsilon_a \text{ or } \left| \frac{\mathbf{k}}{n} - \frac{M}{N} \right| < \varepsilon_r \frac{M}{N} \right\}$  with respect to  $M$  is attained at

$$\begin{aligned} & \left\{ L, U, \left\lfloor \frac{N\varepsilon_a}{\varepsilon_r} \right\rfloor, 1 + \left\lfloor \frac{N\varepsilon_a}{\varepsilon_r} \right\rfloor \right\} \cup \left\{ \left\lfloor N \left( \frac{k}{n} - \varepsilon_a \right) \right\rfloor \in \left( L, \left\lfloor \frac{N\varepsilon_a}{\varepsilon_r} \right\rfloor \right) : k \in \mathbb{Z} \right\} \\ & \cup \left\{ \left\lfloor N \left( \frac{k}{n} + \varepsilon_a \right) \right\rfloor \in \left( L, \left\lfloor \frac{N\varepsilon_a}{\varepsilon_r} \right\rfloor \right) : k \in \mathbb{Z} \right\} \\ & \cup \left\{ \left\lfloor \frac{Nk}{(1 + \varepsilon_r)n} \right\rfloor \in \left( 1 + \left\lfloor \frac{N\varepsilon_a}{\varepsilon_r} \right\rfloor, U \right) : k \in \mathbb{Z} \right\} \\ & \cup \left\{ \left\lfloor \frac{Nk}{(1 - \varepsilon_r)n} \right\rfloor \in \left( 1 + \left\lfloor \frac{N\varepsilon_a}{\varepsilon_r} \right\rfloor, U \right) : k \in \mathbb{Z} \right\}, \end{aligned}$$

which has less than  $2n \left( \frac{U-L-3}{N} \right) + 8$  elements.

It should be noted that Theorem 3 can be shown by applying Theorem 1 and Theorem 2 with the observation that

$$\Pr \left\{ \left| \frac{\mathbf{k}}{n} - \frac{M}{N} \right| < \varepsilon_a \text{ or } \left| \frac{\mathbf{k}}{n} - \frac{M}{N} \right| < \varepsilon_r \frac{M}{N} \right\} = \begin{cases} \Pr \left\{ \left| \frac{\mathbf{k}}{n} - \frac{M}{N} \right| < \varepsilon_a \right\} & \text{for } M \in \left[ L, \left\lfloor \frac{N\varepsilon_a}{\varepsilon_r} \right\rfloor \right], \\ \Pr \left\{ \left| \frac{\mathbf{k}}{n} - \frac{M}{N} \right| < \varepsilon_r \frac{M}{N} \right\} & \text{for } M \in \left[ 1 + \left\lfloor \frac{N\varepsilon_a}{\varepsilon_r} \right\rfloor, U \right]. \end{cases}$$

Finally, we would like to point out that similar characteristics of coverage probability can be shown for other populations such as Bernoulli population and Poisson population, which allows for the exact computation of minimum sample size. For details, see our recent papers [1, 2].

## 5 Conclusion

In this paper, we develop an exact method for the computation of the minimum sample size for estimation of the proportion of finite population. The method is much more efficient than previously possible. The efficiency improvement is due to the interesting discovery of the characteristics of the coverage probability. Such characteristics reveals a new aspect of the hyper-geometrical distribution.

## A Proof of Theorem 1

Define

$$C(M) = \Pr \left\{ \left| \frac{\mathbf{k}}{n} - \frac{M}{N} \right| < \varepsilon \right\} = \Pr \{ g(M) \leq \mathbf{k} \leq h(M) \}$$

where

$$g(M) = \left\lfloor n \left( \frac{M}{N} - \varepsilon \right) \right\rfloor + 1, \quad h(M) = \left\lceil n \left( \frac{M}{N} + \varepsilon \right) \right\rceil - 1.$$

It should be noted that  $C(M)$ ,  $g(M)$  and  $h(M)$  are actually multivariate functions of  $M$ ,  $N$ ,  $\varepsilon$  and  $n$ . For simplicity of notations, we drop the arguments  $n$ ,  $N$  and  $\varepsilon$  throughout the proof of Theorem 1.

**Lemma 1** Let  $0 \leq M < M+1 \leq N$ . Define  $T(k, M, N, n) = \binom{M}{k} \binom{N-M-1}{n-k-1} / \binom{N}{n}$ . Then,  $S(n, 0, k, M, N) - S(n, 0, k, M+1, N) = T(k, M, N, n)$  for any integer  $k$ .

**Proof.** We first show the equation for  $0 \leq k \leq M$ . We perform induction on  $k$ . For  $k = 0$ , we have

$$\begin{aligned}
S(n, 0, k, M, N) - S(n, 0, k, M+1, N) &= S(n, 0, 0, M, N) - S(n, 0, 0, M+1, N) \\
&= \frac{\binom{M}{0} \binom{N-M}{n}}{\binom{N}{n}} - \frac{\binom{M+1}{0} \binom{N-M-1}{n}}{\binom{N}{n}} \\
&= \frac{\binom{N-M-1}{n-1}}{\binom{N}{n}} \\
&= \frac{\binom{M}{0} \binom{N-M-1}{n-0-1}}{\binom{N}{n}} = T(0, M, N, n),
\end{aligned} \tag{1}$$

where (1) follows from the fact that, for non-negative integer  $m$ ,

$$\binom{m+1}{z+1} = \binom{m}{z} + \binom{m}{z+1} \tag{2}$$

for any integer  $z$ .

Now suppose the lemma is true for  $k-1$  with  $1 \leq k \leq M$ , i.e.,

$$S(n, 0, k-1, M, N) - S(n, 0, k-1, M+1, N) = \frac{\binom{M}{k-1} \binom{N-M-1}{n-k}}{\binom{N}{n}}.$$

Then,

$$\begin{aligned}
S(n, 0, k, M, N) - S(n, 0, k, M+1, N) &= S(n, 0, k-1, M, N) - S(n, 0, k-1, M+1, N) \\
&\quad + \frac{\binom{M}{k} \binom{N-M}{n-k}}{\binom{N}{n}} - \frac{\binom{M+1}{k} \binom{N-M-1}{n-k}}{\binom{N}{n}} \\
&= \frac{\binom{M}{k-1} \binom{N-M-1}{n-k}}{\binom{N}{n}} + \frac{\binom{M}{k} \binom{N-M}{n-k}}{\binom{N}{n}} - \frac{\binom{M+1}{k} \binom{N-M-1}{n-k}}{\binom{N}{n}} \\
&= \frac{\binom{M}{k} \binom{N-M}{n-k}}{\binom{N}{n}} - \left[ \frac{\binom{M+1}{k} \binom{N-M-1}{n-k}}{\binom{N}{n}} - \frac{\binom{M}{k-1} \binom{N-M-1}{n-k}}{\binom{N}{n}} \right] \\
&= \frac{\binom{M}{k} \binom{N-M}{n-k}}{\binom{N}{n}} - \frac{\binom{M}{k} \binom{N-M-1}{n-k}}{\binom{N}{n}} \\
&= \frac{\binom{M}{k} \binom{N-M-1}{n-k-1}}{\binom{N}{n}}
\end{aligned} \tag{3}$$

$$= \frac{\binom{M}{k} \binom{N-M-1}{n-k-1}}{\binom{N}{n}} \tag{4}$$

where (3) and (4) follows from (2). Therefore, we have shown the lemma for  $0 \leq k \leq M$ .

For  $k > M$ , we have  $S(n, 0, k, M, N) = S(n, 0, k, M+1, N) = 1$  and  $T(k, M, N, n) = 0$ . For  $k < 0$ , we have  $S(n, 0, k, M, N) = S(n, 0, k, M+1, N) = 0$  and  $T(k, M, N, n) = 0$ . Thus, the lemma is true for any integer  $k$ .

□

**Lemma 2** *Let  $1 \leq M \leq N$  and  $k \leq l$ . Then,*

$$S(n, k, l, M, N) - S(n, k, l, M - 1, N) = T(k - 1, M - 1, N, n) - T(l, M - 1, N, n).$$

**Proof.** To show the lemma, it suffices to consider 6 cases as follows.

Case (i):  $0 < n < k \leq l$ . In this case,  $S(n, k, l, M, N) = S(n, k, l, M - 1, N) = 0$  and  $T(k - 1, M - 1, N, n) = T(l, M - 1, N, n) = 0$ .

Case (ii):  $k \leq l < 0 < n$ . In this case,  $S(n, k, l, M, N) = S(n, k, l, M - 1, N) = 0$  and  $T(k - 1, M - 1, N, n) = T(l, M - 1, N, n) = 0$ .

Case (iii):  $k \leq 0 < n \leq l$ . In this case,  $S(n, k, l, M, N) = S(n, k, l, M - 1, N) = 1$  and  $T(k - 1, M - 1, N, n) = T(l, M - 1, N, n) = 0$ .

Case (iv):  $k \leq 0 \leq l < n$ . In this case,  $T(k - 1, M - 1, N, n) = 0$  and, by Lemma 1,

$$\begin{aligned} S(n, k, l, M, N) - S(n, k, l, M - 1, N) &= [S(n, 0, l, M, N) - S(n, 0, l, M - 1, N)] \\ &= T(k - 1, M - 1, N, n) - T(l, M - 1, N, n). \end{aligned}$$

Case (v):  $0 < k \leq n \leq l$ . In this case,  $T(l, M - 1, N, n) = 0$  and, by Lemma 1,

$$\begin{aligned} S(n, k, l, M, N) - S(n, k, l, M - 1, N) &= [S(n, 0, k - 1, M - 1, N) - S(n, 0, k - 1, M, N)] \\ &= T(k - 1, M - 1, N, n) - T(l, M - 1, N, n). \end{aligned}$$

Case (vi):  $0 < k \leq l < n$ . In this case, by Lemma 1,

$$\begin{aligned} S(n, k, l, M, N) - S(n, k, l, M - 1, N) &= [S(n, 0, l, M, N) - S(n, 0, k - 1, M, N)] \\ &\quad - [S(n, 0, l, M - 1, N) - S(n, 0, k - 1, M - 1, N)] \\ &= [S(n, 0, l, M, N) - S(n, 0, l, M - 1, N)] \\ &\quad - [S(n, 0, k - 1, M, N) - S(n, 0, k - 1, M - 1, N)] \\ &= T(k - 1, M - 1, N, n) - T(l, M - 1, N, n). \end{aligned}$$

□

**Lemma 3** *Let  $l \geq 0$  and  $k < n$ . Then,  $\left\lfloor \frac{nM}{N+1} \right\rfloor \geq l$  for  $M \geq 1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor$ , and  $\left\lfloor \frac{nM}{N+1} \right\rfloor \leq k - 1$  for  $M \leq 1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor$ .*

**Proof.** To show the first part of the lemma, observe that  $(N + 1 - n)l \geq 0$ , by which we can show  $\frac{nNl}{n-1} \geq (N + 1)l$ . Hence,  $n \left( 1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor \right) > \frac{nNl}{n-1} \geq (N + 1)l$ . That is,  $\frac{n}{N+1} \left( 1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor \right) > l$ . It follows that  $\left\lfloor \frac{n}{N+1} \left( 1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor \right) \right\rfloor \geq l$ . Since the floor function is non-decreasing, we have  $\left\lfloor \frac{nM}{N+1} \right\rfloor \geq l$  for  $M \geq 1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor$ .

To prove the second part of the lemma, note that  $(N+1-n)(n-k) > 0$ , from which we can deduce  $1 + \frac{N(k-1)}{n-1} < \frac{(N+1)k}{n}$ . Hence,  $1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor < \frac{(N+1)k}{n}$ , i.e.,  $\frac{n}{N+1} \left(1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor\right) < k$ , leading to  $\left\lfloor \frac{n}{N+1} \left(1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor\right) \right\rfloor \leq k-1$ . Since the floor function is non-decreasing, we have  $\left\lfloor \frac{nM}{N+1} \right\rfloor \leq k-1$  for  $M \leq 1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor$ .  $\square$

**Lemma 4** *Let  $0 \leq r \leq n$ . Then, the following statements hold true.*

(I)

$$T(r-1, M-1, N, n) \leq T(r, M-1, N, n) \quad \text{for} \quad 1 \leq r \leq \left\lfloor \frac{nM}{N+1} \right\rfloor;$$

$$T(r+1, M-1, N, n) \leq T(r, M-1, N, n) \quad \text{for} \quad \left\lfloor \frac{nM}{N+1} \right\rfloor \leq r \leq n-1.$$

(II)

$$T(r, M-2, N, n) \leq T(r, M-1, N, n) \quad \text{for} \quad 1 < M \leq 1 + \left\lfloor \frac{Nr}{n-1} \right\rfloor;$$

$$T(r, M, N, n) \leq T(r, M-1, N, n) \quad \text{for} \quad 1 + \left\lfloor \frac{Nr}{n-1} \right\rfloor \leq M < N.$$

**Proof.** To show statement (I), note that  $T(r, M-1, N, n) = 0$  for  $\min(M-1, n-1) < r \leq n$ . Our calculation shows that

$$\frac{T(r-1, M-1, N, n)}{T(r, M-1, N, n)} = \frac{r}{M-r} \frac{N-M+1-(n-r)}{n-r} \leq 1 \quad \text{for} \quad 1 \leq r \leq \frac{nM}{N+1}$$

and

$$\frac{T(r-1, M-1, N, n)}{T(r, M-1, N, n)} > 1 \quad \text{for} \quad \frac{nM}{N+1} < r \leq \min(M-1, n-1).$$

To show statement (II), note that  $T(r, M-1, N, n) = 0$  for  $1 \leq M < r+1$ , and  $T(r, M-1, N, n) \geq T(r, M-2, N, n) = 0$  for  $M = r+1$ . Direct computation shows that

$$\frac{T(r, M-1, N, n)}{T(r, M-2, N, n)} = \frac{M-1}{M-1-r} \frac{N-M+2-(n-r)}{N-M+1} \geq 1 \quad \text{for} \quad r+1 < M \leq 1 + \frac{Nr}{n-1},$$

and

$$\frac{T(r, M-1, N, n)}{T(r, M-2, N, n)} < 1 \quad \text{for} \quad 1 + \frac{Nr}{n-1} < M \leq N.$$

$\square$

**Lemma 5** *Let  $k \leq l$  and  $0 \leq L \leq U \leq N$ . Then,*

$$\min_{M \in [L, U]} S(n, k, l, M, N) = \min\{S(n, k, l, L, N), S(n, k, l, U, N)\}.$$

**Proof.** To show the lemma, it suffices to consider 6 cases as follows.

Case (i):  $0 < n < k \leq l$ . In this case,  $S(n, k, l, M, N) = 0$  for any  $M \in [L, U]$ .

Case (ii):  $k \leq l < 0 < n$ . In this case,  $S(n, k, l, M, N) = 0$  for any  $M \in [L, U]$ .

Case (iii):  $k \leq 0 < n \leq l$ . In this case,  $S(n, k, l, M, N) = 1$  for any  $M \in [L, U]$ .

Case (iv):  $k \leq 0 \leq l < n$ . In this case,  $S(n, k, l, M, N) = S(n, 0, l, M, N)$  is non-increasing with respect to  $M \in [L, U]$  as can be seen from Lemma 1.

Case (v):  $0 < k \leq n \leq l$ . In this case,  $S(n, k, l, M, N) = 1 - S(n, 0, k - 1, M, N)$  is non-decreasing with respect to  $M \in [L, U]$  as can be seen from Lemma 1.

Clearly, the lemma is true for the above five cases.

Case (vi):  $0 < k \leq l < n$ . Define  $\Delta(k, l, M, N, n) = S(n, k, l, M, N) - S(n, k, l, M - 1, N)$ . By Lemma 2,  $\Delta(k, l, M, N, n) = T(k - 1, M - 1, N, n) - T(l, M - 1, N, n)$ .

Invoking Lemma 3, for  $M \geq 1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor$ , we have that  $\left\lfloor \frac{nM}{N+1} \right\rfloor \geq l$  and thus, by statement (I) of Lemma 4,  $T(r, M - 1, N, n)$  is non-decreasing with respect to  $r \leq l$ . Consequently,  $T(k - 1, M - 1, N, n) \leq T(l, M - 1, N, n)$ , leading to  $\Delta(k, l, M, N, n) \leq 0$  for  $M \geq 1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor$ .

Similarly, applying Lemma 3, for  $M \leq 1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor$ , we have that  $\left\lfloor \frac{nM}{N+1} \right\rfloor \leq k - 1$  and thus, by statement (I) of Lemma 4,  $T(r, M - 1, N, n)$  is non-increasing with respect to  $r \geq k - 1$ . Consequently,  $T(k - 1, M - 1, N, n) \geq T(l, M - 1, N, n)$ , leading to  $\Delta(k, l, M, N, n) \geq 0$  for  $M \leq 1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor$ .

By statement (II) of Lemma 4, for  $1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor \leq M \leq 1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor$ , we have that  $T(l, M - 1, N, n)$  is non-decreasing with respect to  $M$  and that  $T(k - 1, M - 1, N, n)$  is non-increasing with respect to  $M$ . It follows that  $\Delta(k, l, M, N, n)$  is non-increasing with respect to  $M$  in this range. Therefore, there exists an integer  $M^*$  such that  $1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor \leq M^* \leq 1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor$  and that  $\Delta(k, l, M, N, n) \geq 0$  for  $0 \leq M \leq M^*$ , and  $\Delta(k, l, M, N, n) \leq 0$  for  $M^* \leq M \leq N$ . This implies that  $S(n, k, l, M, N)$  is non-decreasing for  $0 \leq M \leq M^*$  and non-increasing for  $M^* \leq M \leq N$ . This concludes the proof of the lemma.  $\square$

**Lemma 6** *Let  $m$  be an integer. Then,*

$$\left\lceil n \left( \frac{m}{N} + \varepsilon \right) \right\rceil = \begin{cases} k & \text{for } m = \lfloor N \left( \frac{k}{n} - \varepsilon \right) \rfloor, \\ k + 1 & \text{for } \lfloor N \left( \frac{k}{n} - \varepsilon \right) \rfloor < m \leq \lfloor N \left( \frac{k+1}{n} - \varepsilon \right) \rfloor. \end{cases}$$

**Proof.** For notational simplicity, let  $r = \lfloor N \left( \frac{k}{n} - \varepsilon \right) \rfloor$  and  $r' = \lfloor N \left( \frac{k+1}{n} - \varepsilon \right) \rfloor$ . By the definition of the floor function, we have  $N \left( \frac{k}{n} - \varepsilon \right) - 1 < r \leq N \left( \frac{k}{n} - \varepsilon \right)$ , which can be written as  $k - \frac{n}{N} < n \left( \frac{r}{N} + \varepsilon \right) \leq k$ . As a result,  $\left\lceil n \left( \frac{r}{N} + \varepsilon \right) \right\rceil = k$  and the lemma is true for  $m = r$ . Similarly,  $\left\lceil n \left( \frac{r'}{N} + \varepsilon \right) \right\rceil = k + 1$  and the lemma is true for  $m = r'$ .

Since  $r$  is an integer, we have  $N \left( \frac{k}{n} - \varepsilon \right) < r + 1 \leq m \leq r' - 1 \leq N \left( \frac{k+1}{n} - \varepsilon \right) - 1$  for  $r < m < r'$ . Hence,  $\frac{k}{n} < \frac{m}{N} + \varepsilon \leq \frac{k+1}{n} - \frac{1}{N}$ . That is,  $k < n \left( \frac{m}{N} + \varepsilon \right) \leq k + 1 - \frac{n}{N}$ , which implies that  $\left\lceil n \left( \frac{m}{N} + \varepsilon \right) \right\rceil = k + 1$  for  $r < m < r'$ . The proof of the lemma is thus completed.



□

**Lemma 7** *Let  $m$  be an integer. Then,*

$$\left\lfloor n \left( \frac{m}{N} - \varepsilon \right) \right\rfloor = \begin{cases} k & \text{for } \lceil N \left( \frac{k}{n} + \varepsilon \right) \rceil \leq m < \lceil N \left( \frac{k+1}{n} + \varepsilon \right) \rceil, \\ k+1 & \text{for } m = \lceil N \left( \frac{k+1}{n} + \varepsilon \right) \rceil. \end{cases}$$

**Proof.** For notational simplicity, let  $r = \lceil N \left( \frac{k}{n} + \varepsilon \right) \rceil$  and  $r' = \lceil N \left( \frac{k+1}{n} + \varepsilon \right) \rceil$ . By the definition of the ceiling function, we have  $N \left( \frac{k}{n} + \varepsilon \right) \leq r < N \left( \frac{k}{n} + \varepsilon \right) + 1$ , which can be written as  $k \leq n \left( \frac{r}{N} - \varepsilon \right) < k + \frac{n}{N}$ . Hence,  $\lfloor n \left( \frac{r}{N} - \varepsilon \right) \rfloor = k$  and the lemma is true for  $m = r$ . Similarly,  $\lfloor n \left( \frac{r'}{N} - \varepsilon \right) \rfloor = k + 1$  and the lemma is true for  $m = r'$ .

Since  $m$  is an integer, we have  $N \left( \frac{k}{n} + \varepsilon \right) \leq r < m \leq r' - 1 < N \left( \frac{k+1}{n} + \varepsilon \right)$  for  $r < m < r'$ . Hence,  $\frac{k}{n} < \frac{m}{N} - \varepsilon < \frac{k+1}{n}$ , or equivalently,  $k < n \left( \frac{m}{N} - \varepsilon \right) < k + 1$ , which implies that  $\lfloor n \left( \frac{m}{N} - \varepsilon \right) \rfloor = k$  for  $r < m < r'$ .

□

**Lemma 8** *Let  $0 \leq \rho < N$  and  $g \leq h$ . Then  $S(n, g, h+1, \rho+1, N) - S(n, g, h, \rho, N) \geq 0$ .*

**Proof.** Note that, by Lemma 2,

$$\begin{aligned} & S(n, g, h+1, \rho+1, N) - S(n, g, h, \rho, N) \\ &= \binom{\rho+1}{h+1} \binom{N-\rho-1}{n-h-1} \bigg/ \binom{N}{n} + S(n, g, h, \rho+1, N) - S(n, g, h, \rho, N) \\ &= \binom{\rho+1}{h+1} \binom{N-\rho-1}{n-h-1} \bigg/ \binom{N}{n} + T(g-1, \rho, N, n) - T(h, \rho, N, n) \\ &= \left[ \binom{\rho+1}{h+1} \binom{N-\rho-1}{n-h-1} - \binom{\rho}{h} \binom{N-\rho-1}{n-h-1} \right] \bigg/ \binom{N}{n} + T(g-1, \rho, N, n) \\ &= \binom{\rho}{h+1} \binom{N-\rho-1}{n-h-1} \bigg/ \binom{N}{n} + T(g-1, \rho, N, n) \geq 0, \end{aligned}$$

where the last equality follows from (2).

□

**Lemma 9** *Let  $0 < \tau \leq N$  and  $g \leq h$ . Then,  $S(n, g-1, h, \tau-1, N) - S(n, g, h, \tau, N) \geq 0$ .*

**Proof.** Note that, by Lemma 2,

$$\begin{aligned}
& S(n, g-1, h, \tau-1, N) - S(n, g, h, \tau, N) \\
&= \binom{\tau-1}{g-1} \binom{N-\tau+1}{n-g+1} \Big/ \binom{N}{n} + S(n, g, h, \tau-1, N) - S(n, g, h, \tau, N) \\
&= \binom{\tau-1}{g-1} \binom{N-\tau+1}{n-g+1} \Big/ \binom{N}{n} + T(h, \tau-1, N, n) - T(g-1, \tau-1, N, n) \\
&= \left[ \binom{\tau-1}{g-1} \binom{N-\tau+1}{n-g+1} - \binom{\tau-1}{g-1} \binom{N-\tau}{n-g} \right] \Big/ \binom{N}{n} + T(h, \tau-1, N, n) \\
&= \binom{\tau-1}{g-1} \binom{N-\tau}{n-g+1} \Big/ \binom{N}{n} + T(h, \tau-1, N, n) \geq 0,
\end{aligned}$$

where the last equality follows from (2). □

**Lemma 10** *Let  $\rho < \tau$  be two consecutive elements of the ascending arrangement of all distinct elements of*

$$\{L, U\} \cup \left\{ \left\lfloor N \left( \frac{k}{n} - \varepsilon \right) \right\rfloor \in (L, U) : k \in \mathbb{Z} \right\} \cup \left\{ \left\lceil N \left( \frac{k}{n} + \varepsilon \right) \right\rceil \in (L, U) : k \in \mathbb{Z} \right\}.$$

*Then,*

$$C(M) = S(n, g(M), h(M), M, N) \geq \min\{S(n, g(\rho), h(\rho), \rho, N), S(n, g(\tau), h(\tau), \tau, N)\}$$

*for  $\rho \leq M \leq \tau$ .*

**Proof.** Since the lemma is obviously true if  $\tau = \rho + 1$ , we may focus on the situation that  $\rho < \tau - 1$ . To show the lemma, it suffices to consider 7 cases as follows.

Case (i):  $\rho = \left\lfloor N \left( \frac{k}{n} - \varepsilon \right) \right\rfloor$  and  $\tau = \left\lceil N \left( \frac{r}{n} + \varepsilon \right) \right\rceil$ . Note that

$$\left\lceil N \left( \frac{r-1}{n} + \varepsilon \right) \right\rceil \leq \left\lfloor N \left( \frac{k}{n} - \varepsilon \right) \right\rfloor = \rho < \tau = \left\lceil N \left( \frac{r}{n} + \varepsilon \right) \right\rceil \leq \left\lfloor N \left( \frac{k+1}{n} - \varepsilon \right) \right\rfloor.$$

By Lemma 7,  $g(\tau-1) = g(\tau) - 1 = r$ . Since  $\rho < \tau - 1$ , by Lemma 6, we have  $h(\tau-1) = h(\tau) = k$ . By Lemma 9, we have  $S(n, g-1, h, \tau-1, N) - S(n, g, h, \tau, N) \geq 0$ . That is,  $C(\tau-1) \geq C(\tau)$ .

By Lemma 6,  $h(\rho+1) = h(\rho) + 1 = k$ . Since  $\rho < \tau - 1$ , by Lemma 7, we have  $g(\rho) = g(\rho+1) = r$ . By Lemma 8, we have  $S(n, g, h+1, \rho+1, N) - S(n, g, h, \rho, N) \geq 0$ . That is,  $C(\rho+1) \geq C(\rho)$ .

By Lemma 5, we have  $C(M) \geq \min\{C(\rho+1), C(\tau-1)\}$  for  $\rho+1 \leq M \leq \tau-1$ . It follows that  $C(M) \geq \min\{C(\rho), C(\tau)\}$  for  $\rho \leq M \leq \tau$ .

Case (ii):  $\rho = \left\lfloor N \left( \frac{k}{n} - \varepsilon \right) \right\rfloor$  and  $\tau = \left\lfloor N \left( \frac{k+1}{n} - \varepsilon \right) \right\rfloor$ . In this case, there exists an integer  $l$  such that

$$\left\lceil N \left( \frac{l}{n} + \varepsilon \right) \right\rceil \leq \left\lfloor N \left( \frac{k}{n} - \varepsilon \right) \right\rfloor = \rho < \tau = \left\lfloor N \left( \frac{k+1}{n} - \varepsilon \right) \right\rfloor \leq \left\lfloor N \left( \frac{l+1}{n} + \varepsilon \right) \right\rfloor.$$

By Lemma 6, we have  $h(\rho) = k - 1$  and  $h(\rho + 1) = k$ . Since  $\rho < \tau - 1$ , by Lemma 7, we have  $g(\rho + 1) = g(\rho) = l + 1$ . By Lemma 8,  $S(n, g, h + 1, \rho + 1, N) - S(n, g, h, \rho, N) \geq 0$ . That is,  $C(\rho + 1) \geq C(\rho)$ .

For the value of  $\tau$ , there are two sub-cases:

(a)  $\tau < \lceil N \left( \frac{l+1}{n} + \varepsilon \right) \rceil$ . In this case, by Lemma 6 and Lemma 7, we have  $g(M) = g(\tau)$  and  $h(M) = h(\tau)$  for  $\rho < M \leq \tau$ . It follows from Lemma 5 that  $C(M) = S(n, g(M), h(M), M, N) \geq \min\{C(\rho + 1), C(\tau)\}$  for  $\rho < M \leq \tau$ . As a result,  $C(M) \geq \min\{C(\rho), C(\tau)\}$  for  $\rho \leq M \leq \tau$ .

(b)  $\tau = \lceil N \left( \frac{l+1}{n} + \varepsilon \right) \rceil$ . This case is the same as Case (i) previously studied.

Case (iii):  $\rho = \lceil N \left( \frac{k}{n} + \varepsilon \right) \rceil$  and  $\tau = \lceil N \left( \frac{k+1}{n} + \varepsilon \right) \rceil$ . In this case, there exists an integer  $l$  such that

$$\left\lfloor N \left( \frac{l}{n} - \varepsilon \right) \right\rfloor \leq \left\lceil N \left( \frac{k}{n} + \varepsilon \right) \right\rceil = \rho < \tau = \left\lceil N \left( \frac{k+1}{n} + \varepsilon \right) \right\rceil \leq \left\lfloor N \left( \frac{l+1}{n} - \varepsilon \right) \right\rfloor.$$

By Lemma 7,  $g(\tau) = k + 2$  and  $g(\tau - 1) = k + 1$ . Since  $\rho < \tau - 1$ , by Lemma 6, we have  $h(\tau - 1) = h(\tau) = l$ . Thus, by Lemma 9, we have  $S(n, g - 1, h, \tau - 1, N) - S(n, g, h, \tau, N) \geq 0$ . That is,  $C(\tau - 1) \geq C(\tau)$ .

For the value of  $\rho$ , there are two sub-cases.

(a):  $\rho > \lfloor N \left( \frac{l}{n} - \varepsilon \right) \rfloor$ . Since  $\rho < \tau - 1$ , by Lemma 7 and Lemma 6, we have  $g(\rho + 1) = g(\rho) = k + 1$  and  $h(\rho + 1) = h(\rho) = l$ . Hence, by Lemma 5,  $C(M) = S(n, g(M), h(M), M, N) \geq \min\{C(\rho), C(\tau - 1)\}$  for  $\rho$  to  $\tau - 1$ .

(b):  $\rho = \lfloor N \left( \frac{l}{n} - \varepsilon \right) \rfloor$ . This case becomes Case (i) previously studied.

Case (iv):  $\rho = \lceil N \left( \frac{k}{n} + \varepsilon \right) \rceil$  and  $\tau = \lfloor N \left( \frac{r}{n} - \varepsilon \right) \rfloor$ . Note that

$$\left\lfloor N \left( \frac{r-1}{n} - \varepsilon \right) \right\rfloor \leq \left\lceil N \left( \frac{k}{n} + \varepsilon \right) \right\rceil = \rho < \tau = \left\lfloor N \left( \frac{r}{n} - \varepsilon \right) \right\rfloor \leq \left\lceil N \left( \frac{k+1}{n} + \varepsilon \right) \right\rceil.$$

If  $\lfloor N \left( \frac{r-1}{n} - \varepsilon \right) \rfloor = \lceil N \left( \frac{k}{n} + \varepsilon \right) \rceil$ , then the case becomes  $\lfloor N \left( \frac{r-1}{n} - \varepsilon \right) \rfloor = \rho < \tau = \lfloor N \left( \frac{r}{n} - \varepsilon \right) \rfloor$ , which has been studied previously in Case (ii).

If  $\lfloor N \left( \frac{r}{n} - \varepsilon \right) \rfloor = \lceil N \left( \frac{k+1}{n} + \varepsilon \right) \rceil$ , then the case becomes  $\lceil N \left( \frac{k}{n} + \varepsilon \right) \rceil = \rho < \tau = \lceil N \left( \frac{k+1}{n} + \varepsilon \right) \rceil$ , which has been studied previously in Case (iii). So, we only need to consider the situation that

$$\left\lfloor N \left( \frac{r-1}{n} - \varepsilon \right) \right\rfloor < \left\lceil N \left( \frac{k}{n} + \varepsilon \right) \right\rceil = \rho < \tau = \left\lfloor N \left( \frac{r}{n} - \varepsilon \right) \right\rfloor < \left\lceil N \left( \frac{k+1}{n} + \varepsilon \right) \right\rceil.$$

Since  $\rho < \tau - 1$ , by Lemma 6 and Lemma 7, we have

$$g(\rho) = g(\rho + 1) = k + 1, \quad h(\rho) = h(\rho + 1) = r - 1,$$

$$h(\tau - 1) = h(\tau) = r - 1, \quad g(\tau - 1) = g(\tau) = k + 1.$$

It follows that  $g(M) = k + 1$  and  $h(M) = r - 1$  for  $\rho \leq M \leq \tau$ . By Lemma 5,  $C(M) \geq \min\{C(\rho), C(\tau)\}$  for  $\rho \leq M \leq \tau$ .

Case (v):  $\rho = L$  and  $\tau \in \mathcal{Q}$  with

$$\mathcal{Q} = \left\{ \left\lfloor N \left( \frac{k}{n} - \varepsilon \right) \right\rfloor : k \in \mathbb{Z} \right\} \cup \left\{ \left\lceil N \left( \frac{k}{n} + \varepsilon \right) \right\rceil : k \in \mathbb{Z} \right\}.$$

The case that  $\rho = L \in \mathcal{Q}$  can be included in previous cases. So we can focus on the case that  $\rho \notin \mathcal{Q}$ . In this case, by Lemma 5,  $C(M) \geq \min\{C(\rho), C(\tau - 1)\}$  for  $\rho \leq M \leq \tau - 1$ . The comparison of  $C(\tau - 1)$  with  $C(\tau)$  is as before.

Case (vi):  $\rho \in \mathcal{Q}$  and  $\tau = U$ . The case that  $\tau = U \in \mathcal{Q}$  can be included in previous cases. So we can focus on the case that  $\tau \notin \mathcal{Q}$ . In this case, by Lemma 5,  $C(M) \geq \min\{C(\rho + 1), C(\tau)\}$  for  $\rho + 1 \leq M \leq \tau$ . The comparison of  $C(\rho + 1)$  with  $C(\rho)$  is as before.

Case (vii):  $\rho = L$  and  $\tau = U$ . The cases that  $L$  or  $U$  belongs to  $\mathcal{Q}$  can be considered as Case (v) or Case (vi). Thus, we can focus on the case that neither  $L$  nor  $U$  belongs to  $\mathcal{Q}$ . Invoking Lemma 5, we have  $C(M) \geq \min\{C(\rho), C(\tau)\}$  for  $\rho \leq M \leq \tau$ .

□

Now we are in position to prove Theorem 1. Clearly, the statement about the coverage probability follows immediately from Lemma 10. It remains to compute the number of elements in the set. Making use of the fact that, for any real number  $x$  and integer  $r$ ,

$$\lfloor x \rfloor < r \iff x < r, \quad \lfloor x \rfloor > r \iff x \geq 1 + r,$$

we have

$$\left\lfloor N \left( \frac{k}{n} - \varepsilon \right) \right\rfloor \in (L, U) \iff n \left( \frac{L+1}{N} + \varepsilon \right) \leq k < n \left( \frac{U}{N} + \varepsilon \right).$$

That is,

$$\left\lfloor N \left( \frac{k}{n} - \varepsilon \right) \right\rfloor \in (L, U) \iff \left\lceil n \left( \frac{L+1}{N} + \varepsilon \right) \right\rceil \leq k \leq \left\lceil n \left( \frac{U}{N} + \varepsilon \right) \right\rceil - 1.$$

Thus, the number of elements in the set  $\{\lfloor N(\frac{k}{n} - \varepsilon) \rfloor \in (L, U) : k \in \mathbb{Z}\}$  is

$$\left\lceil n \left( \frac{U}{N} + \varepsilon \right) \right\rceil - \left\lceil n \left( \frac{L+1}{N} + \varepsilon \right) \right\rceil < n \left( \frac{U}{N} + \varepsilon \right) - n \left( \frac{L+1}{N} + \varepsilon \right) + 1 = 1 + n \left( \frac{U-L-1}{N} \right),$$

where we have used inequalities  $\lceil x \rceil - \lceil y \rceil < x - \lceil y \rceil + 1 \leq x - y + 1$  for real numbers  $x$  and  $y$ .

Similarly, making use of the fact that, for any real number  $x$  and integer  $r$ ,

$$\lceil x \rceil < r \iff x \leq r - 1, \quad \lceil x \rceil > r \iff x > r,$$

we have

$$\left\lceil N \left( \frac{k}{n} + \varepsilon \right) \right\rceil \in (L, U) \iff n \left( \frac{L}{N} - \varepsilon \right) < k \leq n \left( \frac{U-1}{N} - \varepsilon \right).$$

That is,

$$\left\lceil N \left( \frac{k}{n} + \varepsilon \right) \right\rceil \in (L, U) \iff \left\lfloor n \left( \frac{L}{N} - \varepsilon \right) \right\rfloor + 1 \leq k \leq \left\lfloor n \left( \frac{U-1}{N} - \varepsilon \right) \right\rfloor.$$

Thus, the number of elements in the set  $\{\lceil N(\frac{k}{n} + \varepsilon) \rceil \in (L, U) : k \in \mathbb{Z}\}$  is

$$\left\lfloor n\left(\frac{U-1}{N} - \varepsilon\right) \right\rfloor - \left\lfloor n\left(\frac{L}{N} - \varepsilon\right) \right\rfloor < n\left(\frac{U-1}{N} - \varepsilon\right) - n\left(\frac{L}{N} - \varepsilon\right) + 1 = 1 + n\left(\frac{U-L-1}{N}\right),$$

where we have used inequalities  $\lfloor x \rfloor - \lfloor y \rfloor \leq x - \lfloor y \rfloor < x - y + 1$ .

Therefore, the total number of elements in

$$\{\lfloor N(\frac{k}{n} - \varepsilon) \rfloor \in (L, U) : k \in \mathbb{Z}\} \cup \{\lceil N(\frac{k}{n} + \varepsilon) \rceil \in (L, U) : k \in \mathbb{Z}\}$$

is less than

$$1 + n\left(\frac{U-L-1}{N}\right) + 1 + n\left(\frac{U-L-1}{N}\right) = 2 + 2n\left(\frac{U-L-1}{N}\right).$$

The bound becomes  $4 + 2n\left(\frac{U-L-1}{N}\right)$  since  $L$  and  $U$  are needed to be counted. This concludes the proof of Theorem 1.

## References

- [1] X. Chen, “Exact computation of minimum sample size for estimation of binomial parameters,” arXiv:0707.2113v1 [math.ST], July 2007.
- [2] X. Chen, “Exact computation of minimum sample size for estimation of Poisson parameters,” arXiv:0707.2116v1 [math.ST], July 2007.
- [3] M. M. Desu and D. Raghavarao, *Sample Size Methodology*, Academic Press, 1990.
- [4] S. K. Thompson, *Sampling*, Wiley, 2002.